1. Let $x_1 \geq x_2 \geq ..... \geq 0$. Show that $\sum\limits_1^\infty x_j$ is finite if and only if $\sum\limits_0^\infty y_j < \infty$ where $y_k = 2^k x_{2^k}$ for $k = 0, 1, 2, 3....$. [4]

   Solution: Theorem 3.27 in *Principles of Mathematical Analysis* by *Walter Rudin* $\square$

2. Let $f : [a, b] \to \mathbb{R}$ be continuous and twice differentiable. Show that there exists c in (a, b) such that $f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!} f''(c)$ [5]

   Solution: Put $n = 2$ in Theorem 5.15 in *Principles of Mathematical Analysis* by *Walter Rudin* $\square$

3. Let $y_1, y_2, y_3...$ be any Cauchy sequence of reals. Without using the completeness of $\mathbb{R}$, show that the sequence $y_1, y_2...$ is a bounded sequence. [2]

   Solution: Let $\epsilon > 0$. There exists $N \in \mathbb{N}$ such that $|y_N - y_m| < \epsilon$, $\forall m \geq N$. Therefore every element of the sequence after the N-th stage lies in the closed and bounded interval $[y_N - \epsilon, y_N + \epsilon]$. There are only finitely many elements of the sequence outside this bounded interval. Hence boundedness of the sequence follows. $\square$

4. Show that the complex numbers $\mathbb{C}$ is complete. [You can use $\mathbb{R}$ is complete] [3]

   Solution: Let $\{z_n\}$ be a Cauchy sequence in $\mathbb{C}$ and $z_n = a_n + ib_n$, $\forall n \in \mathbb{N}$. Then $\{a_n\}$ and $\{b_n\}$ are Cauchy in $\mathbb{R}$. Let $\{a_n\}$ converges to a and $\{b_n\}$ converges to b in $\mathbb{R}$ then $\{z_n\}$ converges to $z = a + ib$ in $\mathbb{C}$. $\square$

5. Let $\{x_n\}_{n=1}^\infty$ be a sequence defined by $x_1 = \frac{1}{2}$ and, for any $n \geq 1$,

$$x_{n+1} = \frac{x_n^2}{x_n^2 - x_n + 1}$$

   prove that $\sum_{n=1}^\infty x_n$ is convergent. [5]

   Solution: Consider the function $f(x) = \frac{x}{x^2-x+1} = \frac{x^2}{(x-1)^2+x}$. Hence for $0 < x < 1$, $f(x) < \frac{x^2}{x} = x$. Since $x_1 = \frac{1}{2} < 1$, we get $x_{n+1} = f(x_n) < x_n < 1, \forall n \in \mathbb{N}$, inductively. Thus $\{x_n\}$ is monotonically decreasing. We will apply ratio test to conclude the convergence. $\frac{x_{n+1}}{x_n} = \frac{\frac{x_n^2}{x_n^2 - x_n + 1}}{x_n} = \frac{x_n}{x_n^2 - x_n + 1} = \frac{x_n}{(x_n-1)^2 + x_n} < 1$. The last inequality follows because $x_n - 1 > 0$. $\square$

6. (a) Let u_n be a sequence of complex numbers with $\sum |u_n| < \infty$. Show that $\sum\limits_1^\infty u_n^2$ exists. [3]

   (b) Give an example $a_1, a_2, ...$ a sequence of real numbers such that $\sum\limits_1^\infty a_n$ exists but $\sum a_n^2 = \infty$ and prove your claim. [2]

   Solution:

(a) Since $\sum |u_n| < \infty$, $|u_n| < 1$, for all $n > N$, for some $N \in \mathbb{N}$. Therefore $|u_n|^2 < |u_n|$ for all $n > N$. Hence $\sum\limits_{1}^{\infty} u_n^2$ is absolutely convergent.

(b) Take $a_n = (-1)^n \frac{1}{\sqrt{n}}$. Theorem 3.28 and Theorem 3.43 in *Principles of Mathematical Analysis* by *Walter Rudin* proves what is the required.

$\square$

7. Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying

$$f(x+y) = f(x) + f(y)$$

for all x, y in $\mathbb{R}$. If f is continuous at x_0, show that f is continuous on the whole of $\mathbb{R}$. [3]

Solution: It is easy to see that $f(0) = 0$ and $f(-y) = -f(y), \forall y \in \mathbb{R}$. Since $|f(x) - f(y)| = |f(x-y)|$, f is continuous everywhere once it is continuous at 0. Now since f is continuous at x_0, given $\epsilon > 0$, there exists $\delta > 0$ such that $\big|f(x) - f(x_0)\big| < \epsilon$, whenever $|x - x_0| < \delta$. Therefore, $|f(x - x_0)| < \epsilon$, whenever $|x - x_0| < \delta$. This is nothing but the continuity at 0. $\square$

8. (a) Let $f : [0,1] \to [0,1]$ be continuous. By considering the function $g(x) = f(x) - x$ or otherwise show that there exists x_0 with $f(x_0) = x_0$ [1]

   (b) Let f be as above and satisfying $f(f(y)) = f(y)$ for all y. Let $E_f = \{x : f(x) = x\}$. If E_f has at least two points then show that it must be an interval. [3]

Solution: (a) Assume $f(0) \neq 0$ and $f(1) \neq 1$, we are done otherwise. Consider $g(x) = f(x) - x$, then $g(0) < 0$ and $g(1) > 0$. Hence by intermediate value theorem there exists $x_0 \in [0,1]$ such that $g(x_0) = 0$, hence $f(x_0) = x_0$.

(b) Note that $E_f = \text{Range}(f)$. Since f is continuous on $[0,1]$, range of f is an interval if it is not singleton. $\square$

9. Let $f : [0,1] \to \mathbb{R}$ be continuous in $[0,1]$ and differentiable in $(0,1)$ such that $f(0) = 0$ and $0 \leq f'(x) \leq 2f(x)$, for all $x \in (0,1)$. Prove that $f(x) = 0$ for all $x \in [0,1]$. [Hint: $g(x) = e^{-2x} f(x)$ may be useful.] [3]

Solution: Since $0 \leq f'(x)$, for all $x \in (0,1)$, f is an increasing function on $(0,1)$. Now since $f(0) = 0$, $f \geq 0$. Let us define $g(x) = e^{-2x} f(x)$ on $[0,1]$. Then $g'(x) = e^{-2x}(f'(x) - 2f(x)) \leq 0$ due to the given condition. Hence g is decreasing. But $g(0) = 0$. This implies $g \leq 0$ on $(0,1)$ which implies $f \leq 0$. Therefore, $f = 0$. $\square$

10. Show that if f is continuous on $[0, \infty)$ and uniformly continuous on $[a, \infty)$ for some positive constant a, then f is uniformly continuous on $[0, \infty)$. [4]

Solution: Since f is continuous on $[0,a]$, f is uniformly continuous here. Let $\epsilon > 0$ be given. Then there exists positive δ_1, δ_2 such that

$$|f(x) - f(y)| < \frac{\epsilon}{2} < \epsilon, \tag{1}$$

whenever $x, y \in [0,a]$ and $|x - y| < \delta_1$ OR $x, y \in [a, \infty]$ and $|x - y| < \delta_2$. Take $\delta = \min(\delta_1, \delta_2)$. Let $x \in [0,a], y \in [a, \infty]$ and $|x - y| < \delta$. Then $|f(x) - f(y)| \leq |f(x) - f(a)| + |f(a) - f(y)| < \epsilon$. Hence this δ works for all $x, y \in \mathbb{R}$ and f is uniformly continous on $\mathbb{R}$. $\square$

11. Let $f : [0,1] \to \mathbb{R}$ be a differentiable function such that there is no $x \in [0,1]$ such that $f(x) = f'(x) = 0$. Show that the set $Z := \{x \in [0,1] : f(x) = 0\}$ is finite. [3]

    Solution: Asuume $f(x) = 0$ for infinitely many $x \in [0,1]$. Then there is a limit point $x_0 \in [0,1]$ for this zero set. By continuity $f(x_0) = 0$. We will prove that $f'(x_0) = 0$ to get a contradiction. Let $\{x_n\} \subseteq [0,1]$ be a sequence which converges to x_0 and $f(x_n) = 0, \forall n \in \mathbb{N}$. Then $f'(x_0) = \lim_{n\to\infty} \frac{f(x_n)-f(x_0)}{x_n - x_0} = 0$. $\square$

12. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(r + \frac{1}{n}) = f(r)$ for any rational number r and positive integer n. Prove that f is constant. [Hint: Is $f(r - \frac{1}{n}) = f(r)$ also for rational r and $n = 1, 2, 3...$] [3]

    Solution: Let r be a rational number and $n \in \mathbb{N}$. Then $r - \frac{1}{n}$ is also a rational and $f(r) = f(r - \frac{1}{n} + \frac{1}{n}) = f(r - \frac{1}{n})$. Hence it may be concluded that $f(0) = f(x)$ for any rational x. By continuity we f is a constant. $\square$